Example 1 ($\frac{\partial \tilde{\beta}' A \tilde{\beta}}{\partial \tilde{\beta}} = 2A\tilde{\beta}$)**.** Consider $\tilde{\beta}$ and matrix A as following,

$$\tilde{\beta} = [\tilde{\beta}_1, \tilde{\beta}_2, \dots, \tilde{\beta}_k]', A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \vdots & \dots & \ddots & \vdots \\ a_{k1} & a_{k2} & \dots & a_{kk} \end{bmatrix} \forall i, j, a_{ij} = a_{ji}$$

Then, $\frac{\partial \tilde{\beta}' A \tilde{\beta}}{\partial \tilde{\beta}} = 2A\tilde{\beta}$.

Proof. Notice that

$$\tilde{\beta}' A = [\tilde{\beta}_1, \tilde{\beta}_2, \dots, \tilde{\beta}_k] \times \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \vdots & \dots & \ddots & \vdots \\ a_{k1} & a_{k2} & \dots & a_{kk} \end{bmatrix} = \left[\sum_{i=1}^k \tilde{\beta}_i a_{i1}, \sum_{i=1}^k \tilde{\beta}_i a_{i2}, \dots, \sum_{i=1}^k \tilde{\beta}_i a_{ik} \right]$$

Hence,

$$\begin{aligned} \tilde{\beta}' A \tilde{\beta} &= \left[\sum_{i=1}^k \tilde{\beta}_i a_{i1}, \sum_{i=1}^k \tilde{\beta}_i a_{i2}, \dots, \sum_{i=1}^k \tilde{\beta}_i a_{ik} \right] \begin{bmatrix} \tilde{\beta}_1 \\ \tilde{\beta}_2 \\ \vdots \\ \tilde{\beta}_k \end{bmatrix} \\ &= (\tilde{\beta}_1^{\,2} a_{11} + \tilde{\beta}_2^{\,2} a_{22} + \dots + \tilde{\beta}_k^{\,2} a_{kk}) + (\sum_{i \neq 1}^k \tilde{\beta}_i a_{i1}) \tilde{\beta}_1 + (\sum_{i \neq 2}^k \tilde{\beta}_i a_{i2}) \tilde{\beta}_2 + \dots + (\sum_{i \neq k}^k \tilde{\beta}_i a_{ik}) \tilde{\beta}_k \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\partial \tilde{\beta}' A \tilde{\beta}}{\partial \tilde{\beta}_j} &= 2\tilde{\beta}_j a_{jj} + \tilde{\beta}_1 a_{j1} + \tilde{\beta}_2 a_{j2} + \dots + \tilde{\beta}_{j-1} a_{j,j-1} + \sum_{i \neq j}^k \tilde{\beta}_i a_{ij} + \dots + \tilde{\beta}_k a_{jk} \\ &= 2\tilde{\beta}_j a_{jj} + \tilde{\beta}_1 a_{j1} + \tilde{\beta}_2 a_{j2} + \dots + \tilde{\beta}_{j-1} a_{j,j-1} + \sum_{i \neq j}^k \tilde{\beta}_i a_{ji} + \dots + \tilde{\beta}_k a_{jk} \\ &= 2(\tilde{\beta}_1 a_{j1} + \tilde{\beta}_2 a_{j2} + \dots + \tilde{\beta}_k a_{jk}) = 2[a_{j1}, a_{j2}, \dots, a_{jk}] \begin{bmatrix} \tilde{\beta}_1 \\ \tilde{\beta}_2 \\ \vdots \\ \tilde{\beta}_k \end{bmatrix} \end{aligned}$$

□

Example 2 $(\sigma^{-2}(n-K)s_{UB}^2 \sim \chi^2(n-K))$**.** To prove:

$$\sigma^{-2}(n-K)s_{UB}^2 \sim \chi^2(n-K)$$

we need to following two facts:

(i) If random vector $\mathbf{x} \sim N(\mathbf{0}, \mathbf{I}_n)$ and $\mathbf{A}$ is idempotent matrix, then $\mathbf{x}'\mathbf{A}\mathbf{x} \sim \chi^2(\text{rank}(\mathbf{A}))$, where $\text{rank}(\mathbf{A})$ is the degree of freedom of this chi-square distribution.

(ii) For an idempotent matrix M, we have: $\text{rank}(M) = \text{trace}(M)$.

Proof. Assume we have known these two facts, let's begin to prove it:

$$\begin{aligned}
\sigma^{-2}(n-K)s_{UB}^2 &= \frac{\sum_{i=1}^{n}(\hat{\varepsilon_i}^2)}{\sigma^2} \\
&= \frac{\hat{\varepsilon}' \cdot \hat{\varepsilon}}{\sigma^2} \\
&= \frac{(Y-\hat{Y})'(Y-\hat{Y})}{\sigma^2} \\
&= \frac{(Y-X\hat{\beta})'(Y-X\hat{\beta})}{\sigma^2} \\
&= \frac{(Y-X(X'X)^{-1}X'Y)'(Y-X(X'X)^{-1}X'Y)}{\sigma^2} \\
&= \frac{[(I_n-X(X'X)^{-1}X')Y]'[(I_n-X(X'X)^{-1}X')Y]}{\sigma^2}
\end{aligned}$$

Recall that

$$M = I_n - X(X'X)^{-1}X' = I_n - P.$$

Hence,

$$\begin{aligned}
\sigma^{-2}(n-K)s_{UB}^2 &= \frac{(MY)'(MY)}{\sigma^2} \\
&= \frac{[M(X\beta+\varepsilon)]'[M(X\beta+\varepsilon)]}{\sigma^2} \\
&= \frac{[(I_n-P)X\beta+M\varepsilon]'[(I_n-P)X\beta+M\varepsilon]}{\sigma^2}
\end{aligned}$$

Notice that:

$$(I_n-P)X\beta+M\varepsilon = (I_n-X(X'X)^{-1}X')X\beta+M\varepsilon = M\varepsilon$$

Hence,

$$\sigma^{-2}(n-K)s_{UB}^2 = \frac{(M\varepsilon)'(M\varepsilon)}{\sigma^2}$$

M is idempotent implies

$$\sigma^{-2}(n-K)s_{UB}^2 = \frac{\varepsilon' M\varepsilon}{\sigma^2} = \frac{\varepsilon'}{\sigma}M\frac{\varepsilon}{\sigma}$$

By fact (i), we can assert that: $\sigma^{-2}(n-K)s_{UB}^2 \sim \chi^2(\text{rank}(M))$, now we need to show: $\text{rank}(M) = n-K$. By fact (ii), we need to show: $\text{trace}(M) = n-K$.

$$\begin{aligned} trace(M) &= trace(I_n - X(X'X)^{-1}X') \\ &= trace(I_n) - trace(X(X'X)^{-1}X') \end{aligned}$$

As $trace(AB) = trace(BA)$, hence

$$\begin{aligned} trace(M) &= n - trace((X'X)^{-1}X'X) \\ &= n - K \end{aligned}$$

□