A Note on Utility Representations of Lexicographic Preferences

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Abstract

In this note, we provide the condition for lexicographic preferences to admit utility representations. To be specific, for $X_i \subset \mathbb{R}$, the lexicographic preferences admit utility representation on $X_1 \times X_2 \cdots \times X_n$ if and only if $X_1, \ldots, X_{n-1}$ are countable.

Keywords: lexicographic preferences; utility representation

1 Introduction

Lexicographic preferences derive from reality. A typical example is the Chinese College Application. The student should first choose the order of universities and then decide their desiring majors. Formally, we give the definition of lexicographic preferences.

Definition 1 (Lexicographic preference)**.** Consider a binary relationship $\succsim$ on $X_1 \times \cdots \times X_n$ where $X_i \subset \mathbb{R}$ for $i = 1, \ldots, n$. Denote $X = X_1 \times \cdots \times X_n$. $\succsim$ is a lexicographic preference if for $x, y \in X$,

$$x \sim y, \text{ if } x = y$$
$$x \succ y, \text{ if there is } i \in \{1, \ldots, n\} \text{ such that } x_j = y_j \text{ for all } j < i \text{ and } x_i > y_i$$

The following two examples illustrate utility representations on restricted domains.

Example 1 (Lexicographic preference on $\mathbb{N} \times \mathbb{R}$)**.** Consider lexicographic preference on $\mathbb{N} \times \mathbb{R}$, then it has utility function

$$u(x_1, y_1) = x_1 + \frac{\arctan(y_1)}{\pi}.$$

Proof. Consider pairs (x_1, y_1) and (x_2, y_2) with $x_2 > x_1$. Then,

$$\begin{aligned} u(x_2, y_2) - u(x_1, y_1) &= x_2 + \arctan(y_2) - x_1 - \arctan(y_1) \\ &\geq x_2 - \frac{\frac{\pi}{2}}{\pi} - x_1 - \frac{\frac{\pi}{2}}{\pi} \\ &= x_2 - x_1 - 1 \geq 0. \end{aligned}$$

□

Example 2 (Lexicographic preference on more fancy setting)**.** Consider lexicographic preference on $([0,1]\cap\mathbb{Q})\times[0,1]$, then for all $(x,y)\in([0,1]\cap\mathbb{Q})\times[0,1]$ it has utility function

$$u(x,y)=\sum_{k=1}^{l-1}2^{-k}+\frac{2^{-l}}{2}y,\ \text{where } x^l=x.$$

where $[0,1]\cap\mathbb{Q}=\{x^1,x^2,\dots\}$.

Proof. We discuss it into three cases.

(i) if $(x_1,y_1)=(x_2,y_2)$, then $u(x_1,y_1)=u(x_2,y_2)$;

(ii) if $x_1>x_2$: let $x^{l_1}=x_1$ and $x^{l_2}=x_2$.

$$u(x_1,y_1)\geq u(x_1,0)\geq\sum_{k=1}^{l_2}2^{-k}$$
$$u(x_2,y_2)\leq u(x_2,1)=\sum_{k=1}^{l_2-1}2^{-k}+2^{-l_2-1}$$

clearly, $u(x_1,y_1)\geq u(x_2,y_2)$.

(iii) if $x_1=x_2$ and $y_1>y_2$, the conclusion is obvious.

□

These two examples potentially indicate that for $A\subset\mathbb{R}$ and $B\subset\mathbb{R}$, lexicographic preferences on $A\times B$ might admit a utility representation if A is countable. The next theorem verifies our conjecture.

2 Result and Intuition

Theorem 1. Consider $X_1,\dots,X_n\subset\mathbb{R}$ and each X_i is infinite. Lexicographic preferences on $X_1\times\cdots\times X_n$ admit utility representation if and only if $X_1,\dots,X_{n-1}$ are countable.

Proof. First, we prove "only if", then we prove "if".

(i) "$\rightarrow$": proof by contradiction and assume $\succsim$ admits a utility representation. Without loss of generality, suppose $X_i\in\{X_1,\dots,X_{n-1}\}$ is uncountable. Then, consider

$$(\bar{x}_1,\dots,\bar{x}_i,a,\dots,\bar{x}_n),\ (\bar{x}_1,\dots,\bar{x}_i,b,\dots,\bar{x}_n),\quad a,b\in X_{i+1}\text{ and } a>b.$$

Hence,

$$u(\bar{x}_1,\dots,\bar{x}_i,a,\dots,\bar{x}_n)>u(\bar{x}_1,\dots,\bar{x}_i,b,\dots,\bar{x}_n).$$

Pick one rational number from interval

$$q(\bar{x}_i)\in[u(\bar{x}_1,\dots,\bar{x}_i,b,\dots,\bar{x}_n),u(\bar{x}_1,\dots,\bar{x}_i,a,\dots,\bar{x}_n)]$$

Further notice that for $\bar{x}_i' > \bar{x}_i$, we have

$$\begin{aligned} q(\bar{x}_i') &\geq u(\bar{x}_1, \ldots, \bar{x}_i', b, \ldots, \bar{x}_n) \\ &> u(\bar{x}_1, \ldots, \bar{x}_i, a, \ldots, \bar{x}_n) \\ &\geq q(\bar{x}_i). \end{aligned}$$

Consider a mapping from X_i to the set of all $q(x_i)$, where $x_i \in X_i$. It is injection by above argument and surjection by our construction of codomain. Hence, we form a bijection from uncountable set to a countable set[1], contradiction yields.

(ii) "$\leftarrow$": without loss of generality, we can assume $X_n = \mathbb{R}$. The reason is following: if we can find utility representation on $X_1 \times \cdots \times \mathbb{R}$, then this utility representation also preserves order on $X_1 \times \cdots \times X_n$, when $X_n \subset \mathbb{R}$. Let

$$X = X_1 \times \cdots \times X_{n-1} \times \mathbb{R}.$$

Consider set $S = X_1 \times \cdots \times X_{n-1} \times \mathbb{Q}$. S is countable and we would like to show S is separable set for $\succsim$. Suppose $x, y \in X \backslash S$ and $x \succ y$. There exists $i \in \{1, \ldots, n\}$ such that

$$x_j = y_j \text{ for all } j < i \text{ and } x_i > y_i.$$

We can find out $z \in S$ such that $x \succ z \succ y$ by following way:

(a) If there is $z_i \in X_i$ such that $x_i > z_i > y_i$, then we are done. Let $z_j = x_j = y_j$ for all $j < i$ and $z_i = z_i$, and for $k > i$ the value of z_k does not matter;

(b) If there is not $z_i \in X_i$ such that $x_i > z_i > {y_i}^2$, let $z_j = y_j$ for all $j \leq i$.

   i. Now for z_{i+1}, if there is $z_{i+1} \in X_{i+1}$ such that $z_{i+1} > y_{i+1}$, we are done.

   ii. If not, let $z_{i+1} = y_{i+1}$.

   iii. Continue above process until we find some $z_k > y_k$ for $k > i+1$.Finally we come to find an element in $z_n \in X_n = \mathbb{Q}$ such that $z_n > y_n$. Such z_n's existence can be guaranteed since we can always find $z_n \in (y_n, y_n+1) \cap \mathbb{Q}$.

In this way, we construct z to separate between x and y. Finally, since $\succsim$ on $X_1 \times \cdots \times \mathbb{R}$ is complete, transitive, and separable, it admits a utility representation. This utility representation preserves order on $X_1 \times \cdots \times X_n$.

$\square$

Intuition 1. Think of $\succsim$ on $\mathbb{N} \times \mathbb{R}$. For any pairs (x_1, x_2) and (y_1, y_2), we can assign a greater weight to the first coordinate and a less weight to the second coordinate. Notice that the least gap between x_1, y_1 is 1, hence let the maximal gap between x_2, y_2 less than 1 will do the trick. Then, an ideal expression will be

$$u(x_1, x_2) = x_1 + \frac{\arctan(x_2)}{\pi}.$$

[1] Infinite subset of a countable set is still countable.

[2] This happens when X_i has gap.