Example 1 (CES utility function)**.** The constant elasticity of substitute utility function has form of,

$$U(x) = (\sum_{i=1}^{n} x_i^{\rho})^{\frac{1}{\rho}}.$$

The constant elasticity is indeed: $\frac{1}{1-\rho}$, before we proceed to proof notice that

$$MRS_{ji} = \frac{\frac{1}{\rho}(\sum_{i=1}^{n} x_i^{\rho})^{\frac{1-\rho}{\rho}} \rho x_j^{\rho-1}}{\frac{1}{\rho}(\sum_{i=1}^{n} x_i^{\rho})^{\frac{1-\rho}{\rho}} \rho x_i^{\rho-1}} = (\frac{x_j}{x_i})^{\rho-1}$$

Hence,

$$\begin{aligned} E_{ij} &= \frac{\partial ln(\frac{x_i}{x_j})}{\partial ln(MRS_{ji})} \\ &= \frac{\partial ln(\frac{x_i}{x_j})}{\partial ln[(\frac{x_i}{x_j})^{1-\rho}]} \\ &= \frac{1}{1-\rho}. \end{aligned}$$

Example 2 (CES production function)**.** The CES production function is given by,

$$f(x) = A(\sum_{i=1}^{n} \lambda_i x_i^{\rho})^{\frac{k}{\rho}}.$$

More often, we focus on $f(x) = (\sum_{i=1}^{n} a_i x_i^{\gamma})^{\frac{1}{\gamma}}$ with $\sum_{i=1}^{n} a_i = 1$ and this form has following three "transformations" according to different cases of γs.

(i) $\lim_{r\to 1}(\sum_{i=1}^{n} a_i x_i^{\gamma})^{\frac{1}{\gamma}} = \sum_{i=1}^{n} a_i x_i$: this is obvious;

(ii) $\lim_{r\to 0}(\sum_{i=1}^{n} a_i x_i^{\gamma})^{\frac{1}{\gamma}} = \prod_{i=1}^{n} x_i^{a_i}$:

Proof.

$$\lim_{r\to 0}(\sum_{i=1}^{n} a_i x_i^{\gamma})^{\frac{1}{\gamma}} = \lim_{r\to 0} e^{\frac{1}{\gamma} ln(\sum_{i=1}^{n} a_i x_i^{\gamma})}$$

Notice that L'Hôpital's Rule implies,

$$\lim_{r\to 0} \frac{ln(\sum_{i=1}^{n} a_i x_i^{\gamma})}{\gamma} \to$$

$$e^{\sum_{i=1}^{n} a_i ln(x_i)} = \prod_{i=1}^{n} x_i^{a_i}.$$

□

Example 3 (Shadow price)**.** Lagrange multipliers is interpreted as the change in the objective function by relaxing the constraint by one unit, in economics that change can be seen as a value or "shadow price". We shall prove this.

The formal idea is below: say, you are facing the following constrained optimization problem:

$$\begin{cases} \max\limits_{(x,y)} & f(x,y) \\ \text{s.t.} & g(x,y) = c \end{cases} \longrightarrow \begin{cases} x^* & = x^*(c) \\ y^* & = y^*(c) \\ \lambda^* & = \lambda^*(c) \end{cases}$$

We briefly sketch the process of Lagrangian method:

(i) Construct the Lagrangian equation: $\mathcal{L}(x,y,\lambda) = f(x,y) + \lambda(c - g(x,y))$

(ii) Take the derivative with regard to x, y, λ, then we get the first order condition:

$$\frac{\partial \mathcal{L}(x^*,y^*,\lambda^*)}{\partial x} = \frac{\partial f(x^*,y^*)}{\partial x} - \lambda^* \frac{\partial g(x^*,y^*)}{\partial x} = 0$$
$$\frac{\partial \mathcal{L}(x^*,y^*,\lambda^*)}{\partial y} = \frac{\partial f(x^*,y^*)}{\partial y} - \lambda^* \frac{\partial g(x^*,y^*)}{\partial y} = 0$$
$$\frac{\partial \mathcal{L}(x^*,y^*,\lambda^*)}{\partial \lambda} = c - g(x^*,y^*) = 0$$

(iii) Solve this system equations and get the results.

Now, you may wonder if we increase c by one unit, how much would $f(x^*,y^*)$ increase? The answer is: λ^*, that's why Lagrange multipliers are called *shadow prices*, i.e., the change in the objective function by relaxing the constraint by one unit. Let's prove it.

Proof. Writing the value function as: $F(c) = f(x^*(c), y^*(c))$, now take derivative with respect to c:

$$\begin{aligned}
\frac{dF(c)}{dc} &= \frac{\partial f(x^*(c), y^*(c))}{\partial x} \cdot \frac{dx^*(c)}{dc} + \frac{\partial f(x^*(c), y^*(c))}{\partial y} \cdot \frac{dy^*(c)}{dc} \\
&= \underbrace{\lambda^*(c) \cdot \frac{\partial g(x^*, y^*)}{\partial x}}_{FOC} \cdot \frac{dx^*(c)}{dc} + \underbrace{\lambda^*(c) \cdot \frac{\partial g(x^*, y^*)}{\partial y}}_{FOC} \cdot \frac{dy^*(c)}{dc} \\
&= \lambda^*(c) \cdot \underbrace{(\frac{\partial g(x^*, y^*)}{\partial x} \cdot \frac{dx^*(c)}{dc} + \frac{\partial g(x^*, y^*)}{\partial y} \cdot \frac{dy^*(c)}{dc})}_{\text{Take differential with } c\text{: } g(x,y)=c \implies 1} \\
&= \lambda^*(c)
\end{aligned}$$

Analogize to the utility maximizing problem, the interpretation of Lagrange multiplier is: when you have one unit more wealth, then your total utility would increase by λ^*. □

Example 4 (Binary relation as a subset of Cartesian product)**.** Consider outcome set $X = \{a, b, c\}$, we can define a binary relation

$$\succsim = \{(a, a), (b, b), (c, c), (a, b), (b, a), (a, c), (b, c)\}.$$

Example 5 (Continuous preference with discontinuous utility function)**.** A continuous preference can be represented by a discontinuous utility function. Consider $X = \mathbb{R}$ and $\succsim$ is "more is better". Clearly $\succsim$ is continuous, however it can be represented as

$$U(x) = \begin{cases} x, & \text{if } x \leq 0, \\ x + 1, & \text{if } x > 0. \end{cases}$$

Example 6 (Implication of local-nonsatiation preference)**.** Consider two bundles x^j and x^k. $x^j = (x^j_1, x^j_2, \ldots, x^j_n)$ and $x^k = (x^k_1, x^k_2, \ldots, x^k_n) \in \mathbb{R}^n_+$. Bundle x^j satisfies this: when facing price vector $p^j = (p^j_1, p^j_2, \ldots, p^j_n)$, the consumer chooses x^j. Bundle x^k is affordable under p^j, i.e., $p^j x^k < p^j x^j$.

Claim: If the decision maker's choices can be rationalized by a complete locally non-satiated preference relation, then it must be the case that $x^j \succ x^k$.

Proof. We will prove it by contradiction. Assume: $x^k \succsim x^j$ and consider

$$\varepsilon = \frac{p^j x^j - p^j x^k}{\sum_{i=1}^{n} p^j_i}.$$

$$
\begin{aligned}
&\implies \sum_{i=1}^{n} p_i^j y_i < \sum_{i=1}^{n} p_i^j (x_i^k + \varepsilon),\ \forall y \in B_\varepsilon(x_k) \\
&\implies \sum_{i=1}^{n} p_i^j y_i < \sum_{i=1}^{n} p_i^j x_i^k + \varepsilon \sum_{i=1}^{n} p_i^j \\
&\implies \sum_{i=1}^{n} p_i^j y_i < \sum_{i=1}^{n} p_i^j x_i^k + \frac{p^j x^j - p^j x^k}{\sum_{i=1}^{n} p_i^j} \sum_{i=1}^{n} p_i^j \\
&\implies p^j y < p^j x^j
\end{aligned}
$$

This means that you find a ball around x^k, such that at price p^j, every $y \in B_\varepsilon(x^k)$ is affordable.

By non-satiation assumption, $\exists y' \in B_\varepsilon(x^k)$, such that $y' \succ x^k \succsim x^j$. This leads to the contradiction that $\succsim$ rationalized choices, since at price p^j bundle x^j is optimal for decision maker. □

Example 7 (Convex preference may not have concave utility representation)**.** Let's consider the preference on $\mathbb{R}$ such that $x \succsim y$ if $x \geq y$ or $y < 0$. This preference is convex but does not have concave utility representation.

Proof. First we prove it's convex then we show it does not have concave utility function.

(i) *The preference is convex:* we want to show $AsGoodAs(y) = \{z | z \succsim y\}$ for any y in $\mathbb{R}$ is convex. y has the following two cases.

  (i) $y \geq 0$: $\forall x, z \succsim y$ and $\alpha \in (0, 1)$, we have $\alpha x + (1 - \alpha)z \geq y$. This implies $\alpha x + (1 - \alpha)z \succsim y \implies (\alpha x + (1 - \alpha)z) \in AsGoodAs(y)$.

  (ii) $y < 0$. Then x, z have the following three cases.

    (i) $x, z \geq 0$: $\alpha x + (1 - \alpha)z \geq 0 > y$. This implies $(\alpha x + (1 - \alpha)z) \in AsgoodAs(y)$

    (ii) $x \geq 0$ and $z < 0$: $\alpha x + (1 - \alpha)z$ is either ≥ 0 or < 0, both cases imply $\alpha x + (1 - \alpha)z \in AsGoodAs(y)$.

    (iii) $x < 0$ and $z < 0$: $\alpha x + (1 - \alpha)z \sim y$, therefore $\alpha x + (1 - \alpha)z \in AsGoodAs(y)$.

(ii) *The utility representation is not concave:* $\{x < 0\}$ is mapped into a flat line and $\{x \geq 0\}$ is mapped into an increasing function, therefore it can not be concave.

□

Example 8 (Monotonic preference does not imply non-giffen good)**.** Consider the following utility function with two commodities:

$$u(x) = \min\{u_1(x), u_2(x)\}, \text{ where } u_1(x) = x_1 + 10, \ u_2(x) = 2(x_1 + x_2)$$

Now consider the following case, consumer's income is $m = 60$, $p_1 = 12$ and p_2 changes from 9 to 4. This graph shows that when price of good 2

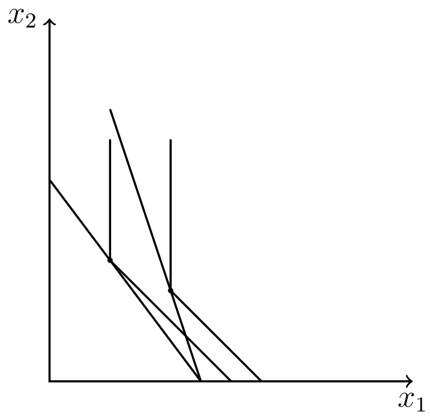

Figure 1: Good 2 is Giffen good

decreases, however, consumer's demand for good 2 decreases as well! Meanwhile, consumer's preferences are *monotonic*!

Example 9 (Turtle traveling)**.** Consider the following two sentences:

(i) The maximal distance a turtle can travel in 2 days is 7 km.

(ii) The minimal time it takes a turtle to travel 7 km is 2 days.

under which conditions are these two sentences equivalent?

(i) For 1 implies 2: we need *monotonicity* of distance-day function. For example, consider the following distance-day function: a turtle can travel 7 km in 1 day but after 1 day it has to rest. Then, this distance-day function is not monotonic and it does not imply 2.

(ii) For 2 implies 1: we need *continuity* of distance-day function. For example, consider a turtle can jump at $t = 2$ from $d = 7$ to $d = 9$, then 1 fails.

Refer to figure **??** for descriptive result.

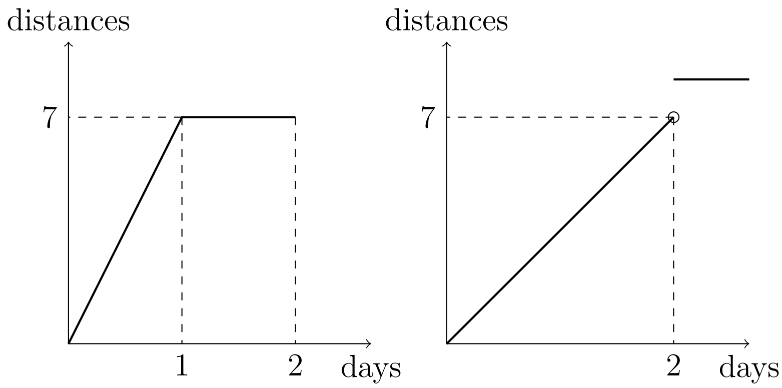

Figure 2: Non-duality: example

Example 10 (Nonsatiated but not monotonic preference)**.** The preference represented by $u(x) = \sqrt{\sum (x_k - x_k^*)^2}$ is non-satiated but not monotonic.

Proof. we want to prove $\forall \varepsilon > 0$, we can find an $x' \in Ball_\varepsilon(x)$, such that $x' \succ x$.

Recall the definition of $\succ$, if I want to find an $x' \in Ball_\varepsilon(x)$, such that $x' \succ x$, I need to show there is one point $x' \in Ball_\varepsilon(x)$, and $d(x, x^*) > d(x', x^*)$. Just imagining in your mind, such x' must exist, because any point x' in $Ball_\varepsilon(x)$ and also it lies on the vector $x^* - x$, then it will have the strict smaller distance with x^* than x.

Based on the above imagination, I construct the x' like this. $(x_1^* - x_1, x_2^* - x_2, \ldots, x_k^* - x_k)$ forms the vector from x^* to x, and I divided the **norm** of this vector to standardize it.

We construct an $x' \in \mathbb{R}_+^K$ like this:

$$x' = (x_1 + \frac{x_1^* - x_1}{d(x^*, x)} \cdot \frac{\varepsilon}{2}, x_2 + \frac{x_2^* - x_2}{d(x^*, x)} \cdot \frac{\varepsilon}{2}, \ldots, x_k + \frac{x_k^* - x_k}{d(x^*, x)} \cdot \frac{\varepsilon}{2})$$

$d(a, b)$ is the Euclidean distance between points a and b.

Without loss of generality, we assume

$$\varepsilon < d(x^*, x)^1 \tag{1}$$

Our proof begins in two steps.

Step1: Prove $x' \succ x$.

[1]Because, any point $x' \in Ball_\varepsilon(x), x' \succ x$, also belongs to $Ball_{\varepsilon'}(x)$, where $\varepsilon' > d(x^*, x)$

Combining with the utility function, this requires us to prove $d(x^*, x') < d(x^*, x)$, let's write down the expression of $d^2(x^*, x')$:

$$
\begin{aligned}
d^2(x^*, x') &= \sum_{i=1}^{k}(x_i + \frac{x_i^* - x_i}{d(x^*, x)} \cdot \frac{\varepsilon}{2} - x_i^*)^2 \\
&= \underbrace{\sum_{i=1}^{k}(x_i - x_i^*)^2}_{=d^2(x^*,x)} + \underbrace{\sum_{i=1}^{k}(\frac{x_i^* - x_i}{d(x^*, x)} \cdot \frac{\varepsilon}{2})^2 - \sum_{i=1}^{k}\frac{(x_i - x_i^*)(x_i - x_i^*)}{d(x^*, x)} \cdot \varepsilon}_{=A} \\
&\longrightarrow A = \sum_{i=1}^{k}(\frac{x_i^* - x_i}{d(x^*, x)} \cdot \frac{\varepsilon}{2})^2 - \sum_{i=1}^{k}\frac{(x_i - x_i^*)(x_i - x_i^*)}{d(x^*, x)} \cdot \varepsilon \\
&\longrightarrow A = \frac{\varepsilon^2}{4} - d(x^*, x) \cdot \varepsilon < \frac{\varepsilon^2}{4} - \varepsilon^2 < 0
\end{aligned}
$$

This completes our proof for step 1.

Step2: Prove $x' \in Ball_\varepsilon(x)$

This proof is trivial. Writing down the distance of x' and x, we will get the result. □