Example 1 (Funny property)**.** If $\cup_{i=1}^{\infty} A_i = M$(universal set), then each $A_i \in \mathcal{A}$.

Proof. Say $B_1, \ldots \notin \mathcal{A}$, whereas $C_1, \cdots \in \mathcal{A}$, then $\cup_{i=1}^{\infty} B_i = M \backslash (\cup_{i=1}^{\infty} C_i) \in \mathcal{A}$. Contradiction yields. $\square$

Example 2 (Non-uniquness of σ-algebra)**.** Notice that for a given set M, the σ-algebra on M is not unique. $\mathcal{P}(M)$ and $\{M, \emptyset\}$ are σ-algebras on M.

Example 3 (Half-interval in 2-dimensional space)**.** This example aims to demonstrate half-interval in a 2-dimensional space. Consider

$$]0,1] : \{x \in \mathbb{R}^2 : 0 < x \leq 1\}.$$

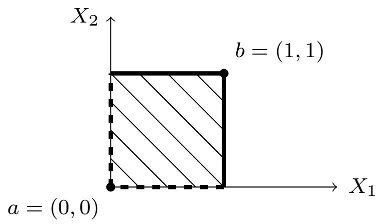

Figure 1: 2-dimension half interval

Example 4. Besides $]a, b]$, there are also other intervals considered to be Borel set.

(i) $\mathbb{R}$ is a Borel set.

Proof. It's clear that $]-\infty, n]$ for $n = 1, 2, \ldots$ is Borel set. Then, from the fact that $\mathcal{B}$ is a σ-algebra on $\mathbb{R}$, we can conclude:

$$\mathbb{R} = \cup_{n=1}^{\infty}]-\infty, n] \in \mathcal{B}.$$

$\square$

(ii) $\{a\}$ for all $a \in \mathbb{R}$ is a Borel set.

Proof. Let $A_n =]a - \frac{1}{n}, a] \in \mathcal{B}$, then $\cup_{n=1}^{\infty} A_n = \{a\} \in \mathcal{B}$. $\square$

(iii) $[a, b]$ for all $-\infty < a < b < \infty$ is a Borel set.

Proof. $(a,b], \{a\}$ are Borel sets, therefore $[a,b] = (a,b] \cup \{a\}$ is a Borel set. $\square$

Example 5 (A π-system needs not to be a σ-algebra.)**.** Let $M = \{1,2\}$. Clearly, $\Big\{\emptyset, \{1\}\Big\}$ is a π-system, but it fails to be a σ-algebra.

Example 6 ($\max\{U,V\}$ can be a simple function)**.** If $U, V \in \mathcal{S}_+$, then $\max\{U,V\}$ and $\min\{U,V\} \in \mathcal{S}_+$.

Proof. Intuitively thinking, $\max\{U,V\}$ should admit finite value and is a nonnegative function as well.

Now we give a formal proof. First, assume $U(x) = \sum\limits_{i=1}^{n} \alpha_i 1_{A_i}$, $V(x) = \sum\limits_{j=1}^{m} \beta_i 1_{B_i}$, where A_i are disjoint and $\cup_{i=1}^{n} A_i = M$, same for B_j. Notice that:

$$\begin{aligned} U(x) &= \sum_{i=1}^{n} \alpha_i 1_{A_i} \\ &= \sum_{i=1}^{n} \alpha_i 1_{\cup_{j=1}^{m}(A_i \cap B_j)} \\ &= \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{m} 1_{A_i \cap B_j} \\ &= \sum_{i,j} \alpha_i 1_{A_i \cap B_j}. \end{aligned}$$

Same argument,

$$V(x) = \sum_{i,j} \beta_j 1_{A_i \cap B_j}$$

Therefore,

$$\max\{f,g\} = \sum_{i,j} \max\{\alpha_i, \beta_j\} 1_{A_i \cap B_j} \in \mathcal{S}_+.$$

$\square$